

# BIANCHI $VI_0$ IN SCALAR AND SCALAR-TENSOR COSMOLOGIES

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## ABSTRACT

We study several cosmological models with Bianchi  $VI_0$  symmetries under the self-similar approach. In order to study how the “constants”  $G$  and  $\Lambda$  may vary, we propose three scenarios where such constants are considered as time functions. The first model is a perfect fluid. We find that the behavior of  $G$  and  $\Lambda$  are related. If  $G$  behaves as a growing time function then  $\Lambda$  is a positive decreasing time function but if  $G$  is decreasing then  $\Lambda$  is negative. For this model we have found a new solution. The second model is a scalar field, where in a phenomenological way, we consider a modification of the Klein-Gordon equation in order to take into account the variation of  $G$ . Our third scenario is a scalar-tensor model. We find three solutions for this models where  $G$  is growing, constant or decreasing and  $\Lambda$  is a positive decreasing function or vanishes. We put special emphasis on calculating the curvature invariants in order to see if the solutions isotropize.

**Keywords:** Scalar and Scalar-Tensor models, Bianchi  $VI_0$ , Self-similar Solutions, Time varying constants

## 1 Introduccion

Current observations of the large scale cosmic microwave background (CMB) suggest to us that our physical universe is expanding in an accelerated way, isotropic and homogeneous models with a positive cosmological constant. The analysis of CMB fluctuations could confirm this picture. But other analyses reveal some inconsistencies. Analysis of WMAP data sets shows us that the universe might have a preferred direction. For this reason, it may be interesting to study Bianchi models since these models may describe such anisotropies.

The observed location of the first acoustic peak of the temperature fluctuations on the CMB corroborated by the data obtained in different experiments [1], indicates that the universe is dominated by an unidentified “dark energy” and suggests that this unidentified dark energy has a negative pressure [2]. This last characteristic of the dark energy points to the vacuum energy or cosmological constant as a possible candidate for dark energy.

In modern cosmological theories, the cosmological constant remains a focal point of interest (see [3]-[6] for reviews of the problem). A wide range of observations now compellingly suggest that the universe possesses a non-zero cosmological constant. Some of the recent discussions on the cosmological constant “problem” and on cosmology with a time-varying cosmological constant point out that in the absence of any interaction with matter or radiation, the cosmological constant remains a “constant”. However, in the presence of interactions with matter or radiation, a solution of Einstein equations and the assumed equation of covariant conservation of stress-energy with a time-varying  $\Lambda$  can be found. This entails that energy has to be conserved by a decrease in the energy density of the vacuum component followed by a corresponding increase in the energy density of matter or radiation. Recent observations strongly favour a significant and a positive value of  $\Lambda$  with magnitude  $\Lambda(G\hbar/c^3) \approx 10^{-123}$ . These observations suggest an accelerating expansion of the universe,  $q < 0$ , [2].

Following Maia, et al [7] who have pointed out that although the cosmological  $\Lambda$ -term has a very small value today, it may contribute to the total energy density of the universe. For this reason, since its present value,  $\Lambda_0$ , may be a remnant of a

primordial inflationary stage, it seems natural to study cosmological scenarios which include a decaying vacuum energy density in such a way that it must be high enough at very early times and sufficiently small at present times in order to be compatible with the current observations. One of the first attempts at considering a decreasing cosmological term was formulated by Chen et al [8]. By studying the Wheeler-DeWitt equation, they argue through dimensional considerations that the cosmological  $\Lambda$ -term must follow a relationship such as  $\Lambda \sim t^{-2}$ , in order to fit with current observations. Other mechanism to describe such variation have been formulated within the framework of the so-called “quintessence models”. Recently this class of models have received a great deal of attention [9] and [10]. Taking into account different observational data it is possible to rule out and to obtain “correct” potential which could play the role of an effective cosmological constant. This strengthens the idea of considering alternative theories where the scalar field is non-minimally coupled to gravity, like scalar-tensor theories (STT) [11]. This class of theories furthermore allows the variation of other constants such as the Newton gravitational one. There are several STT derived from the original one, the Brans-Dicke (BD) model (see for example [12]-[17]). They have been formulated as possible solutions to the discrepancies with observations and try to explain the behaviour of the universe at late times (see [18]-[21]). Of particular interest are the so called chameleon scalar-tensor theories [22].

The study of self-similar (SS) models is quite important since a large class of orthogonal spatially homogeneous models are asymptotically self-similar at the initial singularity and are approximated by exact perfect fluid or vacuum self-similar power-law models. Exact self-similar power-law models can also approximate general Bianchi models at intermediate stages of their evolution. This last point is of particular importance in relating Bianchi models to the real Universe. At the same time, self-similar solutions can describe the behaviour of Bianchi models at late times i.e. as  $t \rightarrow \infty$  (see [23]).

The aim of this work is to study self-similar solutions of a Bianchi VI<sub>0</sub> cosmological model in different contexts and where the “constants”  $G$  and  $\Lambda$  may vary. We are mainly interested in finding exact solutions for the proposed models as well as

to compare the behavior of  $G$  and  $\Lambda$  in the different contexts. In section 2 we start showing all the geometrical ingredients that we are going to use throughout the paper.

We put special emphasis on the study of the curvature invariants in order to study whether the obtained solutions isotropize. Once we have calculated the homothetic vector field (HVF) in section 3 we study the “classical” solution for a vacuum and perfect fluid models comparing these solutions with those obtained ones in a previous work (in that work we used another Bianchi VI<sub>0</sub> metric [24]) as well as a perfect fluid model with time-varying constants. In section 4, we start by studying the kind of potential and scalar fields compatible with self-similar solution. The stated theorems are very general and are valid for all the Bianchi models as well as for the FRW models. All the proofs have been performed by studying the Klein-Gordon equation through the Lie group method. Once we know the scalar fields compatible with the self-similar solution we continue studying a simple scalar model as well as a non-interacting scalar model with matter.

In order to incorporate a gravitational “varying-constant”  $G(t)$  within this framework we purpose, in a phenomenological way, a modified Klein-Gordon equation. As above we need to study the class of potential compatible with a self-similar solution and a varying  $G$ . Two kinds of models are studied. In section 5 we study a generalized scalar-tensor model that determine an accelerated expansion at the present epoch, with arbitrary  $\omega(\phi) = const.$  and  $\Lambda(\phi)$ , where this last function plays the role of an effective cosmological constant.

We would like to emphasize that in order to study the resulting field equations(FE) we have not needed to make any assumption, otherwise, we have deduced, the form of  $\Lambda(\phi)$  by studying the conservation equation through the Lie group method. In section 6, we summarize our results. Finally, in the appendix A, we study through the matter collineation method the kinds of potentials compatible with a self-similar solution in the framework of  $G$  constant. In appendix B we study, using the same method, the  $G$ -var framework in such a way that we regain through this method the results obtained in section 4.

## 2 The Geometric Ingredients

We start by considering the following Killing vector fields (KVF) (see [26])

$$\xi_1 = \partial_x + mz\partial_y + my\partial_z, \quad \xi_2 = \partial_y, \quad \xi_3 = \partial_z, \quad (1)$$

then

$$[\xi_1, \xi_2] = -m\xi_3, \quad [\xi_2, \xi_3] = 0, \quad [\xi_3, \xi_1] = m\xi_2. \quad (2)$$

Note that in this approach is essential to consider the  $m$ -parameter, otherwise it is impossible to obtain self-similar (SS) solutions.

In this way it is obtained the following vector fields  $\{X_j\}$ , such that,  $[\xi_i, X_j] = 0$ ,  $[X_i, X_j] = -C_{ij}^k X_k$  :

$$\begin{aligned} X_1 &= \cosh mx\partial_y + \sinh mx\partial_z, \\ X_2 &= \sinh mx\partial_y + \cosh mx\partial_z, \\ X_3 &= \partial_x, \end{aligned} \quad (3)$$

and the dual 1-forms:

$$\begin{aligned} \omega^1 &= dx \\ \omega^2 &= \cosh mxdy - \sinh mx dz \\ \omega^3 &= -\sinh mxdy + \cosh mx dz, \end{aligned} \quad (4)$$

The metric is defined by

$$ds^2 = -c^2 dt^2 + a^2(t) (\omega^1)^2 + b^2(t) (\omega^2)^2 + d^2(t) (\omega^3)^2 \quad (5)$$

finding that following metric when using Eq. (4)

$$\begin{aligned} ds^2 &= -dt^2 + a^2 dx^2 + (b^2 \cosh^2 mx + d^2 \sinh^2 mx) dy^2 \\ &\quad -2(b^2 + d^2) \cosh mx \sinh mx dy dz + (b^2 \sinh^2 mx \\ &\quad + d^2 \cosh^2 mx) dz^2, \end{aligned} \quad (6)$$

where we have set  $c = 1$ .

We may define the four velocity as follows:  $u^i = (1, 0, 0, 0)$ , in such a way that it is verified,  $g(u^i, u^i) = -1$ . From the definition of the 4-velocity we find that:

$$\begin{aligned} H &= \frac{1}{3} \left( \frac{a'}{a} + \frac{b'}{b} + \frac{d'}{d} \right) = \frac{1}{3} \sum_i H_i, \\ q &= \frac{d}{dt} \left( \frac{3}{H} \right) - 1, \end{aligned} \quad (7)$$

and

$$\sigma^2 = \frac{1}{3} \left( \sum_i H_i^2 - \sum_{i \neq j} H_i H_j \right). \quad (8)$$

Isotropization means, in essence, that at large physical times  $t$ , when the volume factor,  $v = abd$ , tends to infinity, the three scale factors ( $a, b, d$ ) grow at the same rate [27]. We will therefore say, by definition, that a model is isotropizing if, for each scale factors,  $a/f \rightarrow const > 0$ ,  $b/f \rightarrow const > 0$  and  $d/f \rightarrow const > 0$ , as  $v \rightarrow \infty$ , where,  $f = v^{1/3}$ , is the mean scale factor. Then, by rescaling some of the coordinates, we can make  $a/f \rightarrow 1$ ,  $b/f \rightarrow 1$ ,  $d/f \rightarrow 1$  and the metric will become manifestly isotropic at large  $t$ . Two such criteria are  $\mathcal{A} \rightarrow 0$  and  $\sigma \rightarrow 0$ , where the mean anisotropy parameter  $\mathcal{A}$  is defined for the metric as (see, e.g., [28])

$$\mathcal{A} = \frac{\sigma^2}{3H^2} = \frac{1}{3} \sum \frac{H_i^2}{H^2} - 1. \quad (9)$$

The mean anisotropy parameter gives a dimensionless measure of the anisotropy in the Hubble flow by comparing the shear scalar  $\sigma$  to the overall rate of expansion as described by  $H$ . The anisotropy in the temperature of the CMBR enables one to estimate the value of  $\sigma^2$  at the present epoch.

We also study the curvature behaviour of the solutions (see for example [29, 30, 31] and [32]). The studied curvature quantities are the following ones: Ricci scalar,  $I_0 = R^i_i$ , Krestchmann scalar,  $I_1 = R_{ijkl}R^{ijkl}$ , the full contraction of the Ricci tensor,  $I_2 = R_{ij}R^{ij}$ . The Weyl scalar,  $I_3 = C^{abcd}C_{abcd} = I_1 - 2I_2 + \frac{1}{3}I_0^2$ , as well as the electric scalar  $I_4 = E_{ij}E^{ij}$ , [33] and the magnetic scalar  $I_5 = H_{ij}H^{ij}$ , of the Weyl tensor. The Weyl parameter  $\mathcal{W}$  [33], which is a dimensionless measure of the Weyl curvature tensor,

$$\mathcal{W}^2 = \frac{W^2}{H^4} = \frac{1}{6H^4} (E_{ij}E^{ij} + H_{ij}H^{ij}) = \frac{I_3}{24H^4}, \quad (10)$$

can be regarded as describing the intrinsic anisotropy in the gravitational field [34]. Cosmological observations can, in principle, give an upper bound on  $\mathcal{W}$ , although obtaining a strong bound is beyond the reach of present-day observations.

Finally, we shall calculate the gravitational entropy. From a thermodynamic point of view there

is every indication that the entropy of the universe is *increasing*. Increasing gravitational entropy would naturally be reflected by increasing local anisotropy, and the Weyl tensor reflects this. One suggestion in this connection was Penrose’s formulation of what is called the *Weyl curvature conjecture* (WCC) [35]. The hypothesis is motivated by the need for a low entropy constraint on the initial state of the universe when the matter content was in thermal equilibrium. Penrose has argued that the low entropy constraint follows from the existence of the second law of thermodynamics, and that the low entropy in the gravitational field is tied to constraints on the Weyl curvature. Wainwright and Anderson [36] express this conjecture in terms of the ratio of the Weyl and the Ricci curvature invariants,

$$P^2 = \frac{I_3}{I_2}. \quad (11)$$

The physical content of the conjecture is that the initial state of the universe is homogeneous and isotropic. As pointed out by Rothman and Aninos [37, 38] (see also [39]) the entities  $P^2$  and  $I_3$  are “*local*” entities in contrast to what we usually think of entropy. Grøn and Hervik ([30, 31]) have introduced a non-local quantity which shows a more promising behaviour concerning the WCC. This quantity is also constructed in terms of the Weyl tensor, and it has therefore a direct connection with the Weyl curvature tensor but in a “*non-local form*”.

For SS spacetimes, Pelavas and Lake ([40]) have pointed out the idea that Eq. (11) is not an acceptable candidate for gravitational entropy along the homothetic trajectories of any self-similar spacetime. Nor indeed is any “dimensionless” scalar. It is showed that the Lie derivative of any “dimensionless” scalar along a homothetic vector field (HVF) is zero, and concluded that such functions are not acceptable candidates for the gravitational entropy. Nevertheless [41], since self-similar spacetimes represent asymptotic equilibrium states (since they describe the asymptotic properties of more general models), and the result  $P^2 = const.$ , is perhaps consistent with this interpretation since the entropy does not change in these equilibrium models, and perhaps consequently supports the idea that  $P^2$  represents a “gravitational entropy”. As we shall show  $\mathcal{W}^2$  and  $P^2$  will be constant along homothetic trajectories,

since all the dimensionless quantities remain constant along timelike homothetic trajectories.

## 2.1 The homothetic vector field

The homothetic vector field (HVF) is calculated from equation

$$\mathcal{L}_{HO}g_{ij} = 2g_{ij}, \quad (12)$$

(see for example [42]-[46] and [47]). Algebra brings us to obtain the following HVF:

$$\begin{aligned} HO = & (t + t_0) \partial_t + \left(1 - (t + t_0) \frac{a'}{a}\right) x \partial_x \\ & + \left(1 - (t + t_0) \frac{b'}{b}\right) y \partial_y + \left(1 - (t + t_0) \frac{d'}{d}\right) z \partial_z, \end{aligned} \quad (13)$$

with the following constrains for the scale factors:

$$\begin{aligned} a(t) &= a_0 (t + t_0)^{a_1}, \\ b(t) &= b_0 (t + t_0)^{a_2}, \\ d(t) &= d_0 (t + t_0)^{a_3}, \end{aligned} \quad (14)$$

where  $a_1, a_2, a_3 \in \mathbb{R}$ , and the following restrictions for the constants  $a_1, a_2, a_3$  (obtained from the Eq.(12))

$$a_1 = 1, \quad a_2 = a_3. \quad (15)$$

As is observed we have been able to obtain a non-singular solution for the scale factors. Therefore the resulting homothetic vector field is:  $HO = (t + t_0) \partial_t + (1 - a_2) y \partial_y + (1 - a_2) z \partial_z$ .

Since we already know how the scale factors behave, then we may calculate all the curvature invariants as well as all the kinematical quantities.

$$\begin{aligned} H &= \frac{1 + 2a_2}{3(t + t_0)}, \\ q &= 2 \frac{1 - a_2}{2a_2 + 1}, \\ \sigma^2 &= \frac{\sqrt{6} (a_2 - 1)^2}{3 (t + t_0)^2}, \end{aligned} \quad (16)$$

finding that

$$\mathcal{A} = \frac{(a_2 - 1)^2}{(1 + 2a_2)^2} = const., \quad (17)$$

where, as we can see,  $\mathcal{A} \in (0, 1), \forall a_2 \in (0, 1)$ . Although the quantity  $\mathcal{A}$  is constant, this quantity

may take very small values, in fact  $\mathcal{A}$  may runs to zero. It is also observed that the model never inflates since  $q \in (0, 2), \forall a_2 \in (0, 1)$ . Concerning the curvature invariants we find that

$$\begin{aligned} I_0 &= (6a_2^2 - 2m^2) (t + t_0)^{-2}, \\ I_1 &= 4(3(a_2^4 + m^4) - 4a_2^3 + 2a_2^2(1 - m^2) \\ &\quad + 4m^2(a_2 - 1)) (t + t_0)^{-4}, \\ I_2 &= 4(2a_2^2 - 2a_2^3 + 3a_2^4 - 2a_2m^2 + m^4) (t + t_0)^{-4}, \\ I_3 &= 16m^2(m^2 + 6a_2 - 3(1 + a_2^2)) (t + t_0)^{-4} / 3, \\ I_4 &= 2m^4 (t + t_0)^{-4} / 3, \\ I_5 &= 2m^2 (a_2 - 1)^2 (t + t_0)^{-4}, \end{aligned} \quad (18)$$

and

$$\mathcal{W}^2 = \frac{m^2(3(1 + a_2^2) + m^2 - 6a_2)}{9(1 + 2a_2)^4} = const., \quad (19)$$

$$P^2 = \frac{4m^2(m^2 + 6a_2 - 3(1 + a_2^2))}{3(2a_2^2 - 2a_2^3 + 3a_2^4 + (m^2 - 2a_2)m^2)} = const. \quad (20)$$

As above, although  $\mathcal{W}^2 = const \ll 1$ , dimensionless quantity, it may take very small values, for example, if  $a_2 \rightarrow 1$  and  $m \rightarrow 0$  ( $\mathcal{W}^2 \rightarrow m^4/3^6$ ).

## 3 The classical model

We shall take into account the Einstein's field equations (FE) written in the following form:

$$R_{ij} - \frac{1}{2} R g_{ij} = 8\pi G T_{ij}^m - \Lambda g_{ij}, \quad (21)$$

where,  $\Lambda$  is the cosmological constant and  $T_{ij}^m$ , is the energy-momentum tensor defined by

$$T_{ij}^m = (p_m + \rho_m) u_i u_j + p_m g_{ij}, \quad (22)$$

and where the 4-velocity is defined by:  $u_i = (1, 0, 0, 0)$ ,  $\rho_m$  is the energy density and  $p_m$  is the pressure. They are related by the equation:  $p_m = \omega \rho_m$ , with  $\omega \in (-1, 1]$ . In this section we study three models; vacuum solutions, a perfect fluid model and a model with a perfect fluid and where the constants  $G$  and  $\Lambda$  are time varying function.

### 3.1 Vacuum solution

In this case we have found only one solution  $a_2 = m = 0$ . Therefore, the metric Eq. (6) collapses to this one:

$$ds^2 = -dt^2 + (t + t_0)^2 dx^2 + dy^2 + dz^2. \quad (23)$$

We may compare this solution with the one obtained in the paper [24] where we were able to obtain a new solution belonging to Bianchi VI type. In this case, this solution does not belong to Bianchi VI<sub>0</sub> type, so we may say that the metric Eq. (6) is more restrictive than the employed one in [24]. This solution is known as the Taub form of flat space-time ([25] chap. 9).

### 3.2 Perfect Fluid

For this model we obtain the following results:

$$a_2 = \frac{1 - \omega}{2(\omega + 1)} \in (0, 1),$$

$$m = \frac{1}{2} \sqrt{\frac{-(3\omega + 1)(\omega - 1)}{(\omega + 1)^2}} \in \left(0, \frac{1}{2}\right], \quad (24)$$

where  $\omega \in (-\frac{1}{3}, 1)$ , while the scale factors and the energy density behave as

$$a(t) = a_0(t + t_0), \quad d = d_0(t + t_0)^{a_2}, \quad (25)$$

$$b(t) = b_0(t + t_0)^{a_2}, \quad \rho = \rho_0(t + t_0)^{-\gamma},$$

where  $\gamma = (1 + \omega)(1 + 2a_2)$ ,  $\rho_0 = \frac{A}{8\pi G}$ ,  $A = 2a_2 + a_2^2 - m^2$ . With regard to the deceleration parameter

$$q = \frac{1}{2}(1 + 3\omega) > 0$$

$$\mathcal{A} = \frac{(3\omega + 1)^2}{16} = \text{const.} \in (0, 1), \quad (26)$$

$\forall \omega \in (-\frac{1}{3}, 1)$ , so this model does not inflate, and  $\mathcal{A} \rightarrow 0$  only when  $\omega \rightarrow -\frac{1}{3}$ , while the Weyl parameter and the gravitational entropy behaves as

$$\mathcal{W}^2 = -\frac{(3\omega + 1)^2(\omega - 1)(2\omega + 1)}{576} = \text{const} \in (0, 0.012), \quad (27)$$

$$P^2 = \frac{2(3\omega + 1)^2(5\omega + 1)}{3(\omega - 1)(3\omega^2 + 1)} = \text{const.} \in (-\infty, a], \quad a \rightarrow 0^+, \quad (28)$$

where as it is observed,  $\mathcal{W}^2 \leq 0.01$ , it takes a very small values,  $\mathcal{W}^2 \ll 1$ , and it runs to zero if  $\omega \rightarrow 1$  and  $\omega \rightarrow -1/3$ . Notice that our solution is only valid if  $\omega \in (-\frac{1}{3}, 1)$ . Nevertheless  $P^2$  has a very pathological behaviour. For example,  $P^2 \rightarrow a = 2.5000 \times 10^{-11}$  as  $\omega \rightarrow -1/3$ ,  $P^2 = 0$  when  $\omega = -\frac{1}{3}$ , and  $\omega = -\frac{1}{5}$  but  $P^2 \rightarrow -\infty$  when  $\omega \rightarrow 1$ .

Therefore, the metric collapses to this one:

$$ds^2 = -dt^2 + (t + t_0)^2 dx^2 + (t + t_0)^{2a_2} (\cosh 2mxdy^2 - 2 \sinh 2mxdydz + \cosh 2mxdz^2). \quad (29)$$

Note that in [24] we obtain two solutions, while with the metric Eq. (6) we are only able to obtain one solution which coincides with the one obtained in [24]. Therefore, this solution is valid when  $\omega \in (-\frac{1}{3}, 1)$  and  $m \in (0, \frac{1}{2}]$ . It does not accelerate since  $q > 0$ . Nevertheless, we may say that the solution isotropizes since  $\mathcal{A} \rightarrow 0$  when  $\omega \rightarrow -\frac{1}{3}$  (then  $a_2 \rightarrow 1 = a_1$  and  $m \rightarrow 0$ ) and  $\mathcal{W}^2 \ll 1$ . The behaviour of  $P^2$  shows us, that maybe, it is not a good definition for the gravitational entropy (at least in the framework of self-similar solutions) as we have already discussed.

For  $\omega = -\frac{1}{3}$ , and  $\omega = 1$  we get that  $m = 0$ , so the solution does not belong to Bianchi VI<sub>0</sub> type, furthermore if  $\omega = 1$ , then  $a_2 = 0$  i.e. we obtain the vacuum solution.

### 3.3 Time varying constants model

In this framework the FE are the following ones:

$$\frac{a' b'}{a b} + \frac{a' d'}{a d} + \frac{d' b'}{d b} - \left(2 + \frac{b^2}{d^2} + \frac{d^2}{b^2}\right) \frac{m^2}{4a^2} = 8\pi G \rho_m + \Lambda c^2, \quad (30)$$

$$\frac{b''}{b} + \frac{d''}{d} + \frac{d' b'}{d b} + \left(2 + \frac{b^2}{d^2} + \frac{d^2}{b^2}\right) \frac{m^2}{4a^2} = -8\pi G \omega \rho_m + \Lambda c^2, \quad (31)$$

$$\frac{a''}{a} + \frac{b''}{b} + \frac{a' b'}{a b} - \left(2 + \frac{3d^2}{b^2} - \frac{b^2}{d^2}\right) \frac{m^2}{4a^2} = -8\pi G \omega \rho_m + \Lambda c^2, \quad (32)$$

$$\frac{b''}{b} - \frac{d''}{d} + \frac{a' b'}{a b} - \frac{a' d'}{a d} + m^2 \left(\frac{b^2}{d^2 a^2} - \frac{d^2}{b^2 a^2}\right) = 0, \quad (33)$$

$$\frac{d''}{d} + \frac{a''}{a} + \frac{a' d'}{a d} - \left(2 + \frac{3b^2}{d^2} - \frac{d^2}{b^2}\right) \frac{m^2}{4a^2} = -8\pi G \omega \rho_m + \Lambda c^2, \quad (34)$$

$$\rho' + \rho(1 + \omega) \left( \frac{a'}{a} + \frac{b'}{b} + \frac{d'}{d} \right) = 0, \tag{35}$$

$$\Lambda' = -8\pi G' \rho_m, \tag{36}$$

where we have taken into account the condition  $divT = 0$ .

Now, we shall take into account the obtained SS restrictions for the scale factors given by Eq. (15). From Eq. (35) we get

$$\rho = \rho_0 (t + t_0)^{-\gamma}, \tag{37}$$

where  $\gamma = (\omega + 1)h$  and  $h = 1 + 2a_2$ , since  $a_1 = 1$  and  $a_2 = a_3$ .

From Eq. (30) we obtain:

$$\Lambda = \left[ A(t + t_0)^{-2} - 8\pi G\rho_0(t + t_0)^{-(\omega+1)h} \right], \tag{38}$$

where  $A = 2a_2 + a_2^2 - m^2$ . Now, taking into account Eq. (36) and Eq. (38), algebra brings us to obtain

$$G = G_0(t + t_0)^{\gamma-2}, \quad G_0 = \frac{A}{4\pi\rho_0(\omega + 1)}, \tag{39}$$

while the cosmological ‘‘constant’’ behaves as:

$$\Lambda = \frac{A}{c^2} \left( 1 - \frac{2}{\gamma} \right) (t + t_0)^{-2} = \Lambda_0 (t + t_0)^{-2}. \tag{40}$$

In this case we have found the following solution

$$a_{2\pm} = \frac{1}{2} \left( 1 \pm \sqrt{1 - 4m^2} \right), \quad \forall m \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \setminus \{0\}, \tag{41}$$

$a_{2+} \in [1/2, 1)$ ,  $a_{2-} \in [0, 1/2)$ , hence  $h_{\pm} = 1 + 2a_{2\pm}$  and therefore we obtain  $h_+ \in [2, 3)$  and  $h_- \in [1, 2)$

$$\begin{aligned} a(t) &= a_0(t + t_0), & b(t) &= b_0(t + t_0)^{a_{2\pm}}, \\ d &= d_0(t + t_0)^{a_{2\pm}}, & \rho &= \rho_0(t + t_0)^{-\gamma_{\pm}}, \\ G &= G_{0\pm}(t + t_0)^{\gamma_{\pm}-2}, & \Lambda &= \Lambda_{0\pm}(t + t_0)^{-2}, \end{aligned} \tag{42}$$

with  $\gamma_{\pm} = (\omega + 1)h_{\pm}$ . Notice that this solution is valid  $\forall \omega \in (-1, 1]$ . In this way the metric collapses to Eq. (29). The behaviour of the ‘‘constants’’ is

the following one:

$$G \approx \begin{cases} \text{decreasing} & \text{if } (\omega + 1)h_{\pm} < 2 \\ \text{constant} & \text{if } (\omega + 1)h_{\pm} = 2 \\ \text{growing} & \text{if } (\omega + 1)h_{\pm} > 2 \end{cases}$$

$$\Lambda_{0\pm} \approx \begin{cases} < 0 & \text{if } (\omega + 1)h_{\pm} < 2 \\ = 0 & \text{if } (\omega + 1)h_{\pm} = 2 \\ > 0 & \text{if } (\omega + 1)h_{\pm} > 2 \end{cases}, \tag{43}$$

where  $h_{\pm} = 1 + 2a_{2\pm} = 2 \pm \sqrt{1 - 4m^2}$ . Note that in [24] we obtained another solution. With regard to the deceleration parameter (for simplicity we have performed all these calculations with  $h_+$  i.e. with  $a_{2+}$ )

$$q_+ = \frac{3}{h_+} - 1 > 0, \quad q \in \left( 0, \frac{1}{2} \right), \quad \forall m \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \setminus \{0\}, \tag{44}$$

$$\mathcal{A} = \frac{1}{4} \frac{(\sqrt{1 - 4m^2} - 1)^2}{(\sqrt{1 - 4m^2} + 2)^2} = \text{const} \in (0, 0.06) \rightarrow 0, \tag{45}$$

$$\mathcal{W}^2 = -\frac{m^2(4m^2 - 3 + 3\sqrt{1 - 4m^2})}{18(2 + \sqrt{1 - 4m^2})} = \text{const} \in (0, 0.0016) \tag{46}$$

$$P^2 = -\frac{4m^2(8m^2 - 3 + 3\sqrt{1 - 4m^2})}{3((6m^2 - 3)\sqrt{1 - 4m^2} + 12m^2 - 8m^4 - 3)} = \text{const}, \tag{47}$$

where  $P^2 \in (-\infty, 0.01)$ ,  $\forall m \in [-\frac{1}{2}, \frac{1}{2}] \setminus \{0\}$ . Therefore this solution is valid  $\forall \omega \in (-1, 1]$  and  $\forall m \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$ . The model does not accelerate but isotropizes since  $\mathcal{W}^2 \rightarrow 0$  as well as  $\mathcal{A} \ll 1$ . With regard to the quantity  $P^2$  it is observed that  $P^2 \rightarrow 0$ ,  $\forall m \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$  and only runs to minus infinity when  $m \rightarrow \pm \frac{1}{2}$ .

## 4 Scalar field model

In this section we are going to study several scalar models. In the first place we study which kinds of potentials are compatible with the self-similar solution. For this purpose we study through the Lie

group method the resulting Klein-Gordon equation. Once we have deduced the potential compatible with the self-similar solution we study if this kind of potential brings us to obtain self-similar solution. We answer in this case is no, we only obtain power law solutions but this fact does not mean that they are self-similar solution. Therefore, after this brief analysis on the potential, we start by studying a simple scalar model. In second place we study a non-interacting scalar and matter model. We leave for a forthcoming paper the study of the very interesting case of interacting scalar and matter models (see for example [48] and [49]). In the third class of studied models we introduce the hypothesis of a  $G$ -var, i.e. we study a scalar model where  $G = G(t)$ , is a function of time. In this case, in a phenomenological way, we outline a modified Klein-Gordon equation in order to take into account the possible variation on time of the function  $G(t)$ . We go next to study the kind of potential compatible with a self-similar solution and  $G(t)$ . To end this section, we study a model with scalar and matter fields and  $G$ -varying.

The stress-energy tensor may be written in the following form:

$$T_{ij}^\phi = (p^\phi + \rho^\phi) u_i u_j + p^\phi g_{ij}, \quad (48)$$

where the energy density and the pressure of the fluid due a scalar field are given by

$$\rho^\phi = \frac{1}{2} \phi'^2 + V(\phi), \quad p^\phi = \frac{1}{2} \phi'^2 - V(\phi), \quad (49)$$

while the conservation equation reads (Klein-Gordon equation)

$$\phi'' + \phi' H + \frac{d}{d\phi} V = 0. \quad (50)$$

We need to study the class of potential compatible with the SS solution, for this reason we study by using the Lie group method the KG equation, (for an introduction to the Lie group method see for example [50]-[53] and [54] for a concrete application in cosmological contexts). In particular, we seek the forms of  $V(\phi)$  for which our field equations admit symmetries and therefore they are integrable. In this case we already know that the Hubble function behaves as:  $H = h(t + t_0)^{-1}$ ,  $h \in \mathbb{R}^+$ , so the KG equation reads

$$\phi'' + h\phi'(t + t_0)^{-1} + \frac{dV}{d\phi} = 0. \quad (51)$$

**Theorem 1** *The only possible form for the potential  $V(\phi)$  for a spacetime admitting a HFV, HO, is  $V(\phi) = V_0 \exp(\kappa\phi)$  and therefore  $\phi = \alpha \ln t$ .*

**Proof.** The application of the Lie group method brings us to outline the following system of PDEs

$$\xi_{\phi\phi} t^2 = 0, \quad (52)$$

$$2ht\xi_\phi + t^2\eta_{\phi\phi} - 2t^2\xi_{t\phi} = 0, \quad (53)$$

$$3t^2\xi_\phi \frac{dV}{d\phi} + ht\xi_t - h\xi + 2t^2\eta_{t\phi} - t^2\xi_{tt} = 0, \quad (54)$$

$$t^2\eta \frac{d^2V}{d\phi^2} + t^2\eta_{tt} + 2t^2\xi_t \frac{dV}{d\phi} - t^2\eta_\phi \frac{dV}{d\phi} + 3at\eta_t = 0, \quad (55)$$

If we impose the symmetry  $\xi = \alpha(t + t_0)$ ,  $\eta = \delta$ , then its invariant solution is:  $\phi = \frac{\delta}{\alpha} \ln \frac{1}{\alpha}(t + t_0)$ , then, from Eq. (55), we obtain the next restriction for the potential  $V$

$$\delta \frac{d^2V}{d\phi^2} + 2\alpha \frac{dV}{d\phi} = 0, \quad (56)$$

$$V = \beta \exp\left(-2\frac{\alpha}{\delta}\phi\right) + \kappa, \quad \alpha, \beta, \delta, \kappa \in \mathbb{R}.$$

Therefore we have found, redefining the numerical constants, that the only solution compatible with the FE is

$$\phi = \pm\sqrt{\alpha} \ln(t + t_0), \quad V = \beta \exp\left(\mp\frac{2}{\sqrt{\alpha}}\phi\right). \quad (57)$$

as it is required. ■

Note that in this case it is possible to find more symmetries, but the solution generated by them are not compatible with the FE. For example, if we impose the symmetry,  $\xi = \alpha t$ ,  $\eta = \delta\phi$ , then Eq. (55) yields

$$\delta\phi \frac{d^2V}{d\phi^2} + (2\alpha - \delta) \frac{dV}{d\phi} = 0 \Leftrightarrow V = \kappa_1 \phi^{-\frac{2}{\delta}(\alpha - \delta)} + \kappa_2, \quad (58)$$

which is the potential proposed by Peebles and Ratra,  $V \approx \phi^{-\alpha}$  [9], but this solution it is not compatible with the FE with a SS solution. Nevertheless we shall use this potential in the  $G$ -varying scenario. In the appendix we give an alternative derivation of all these results by using the matter collineation approach following a previous paper (see [54]). Models with a self-interaction potential with an exponential dependence on the scalar field of the form  $V = \beta \exp(\mp 2\phi)$ , have been the subject of much interest and arise naturally from theories of gravity such as scalar-tensor theories or string theories [55]. Recently, it has been argued



that a scalar field with an exponential potential is a strong candidate for dark matter in spiral galaxies [56] and is consistent with observations of current accelerated expansion of the universe [57].

In the inverse way we may state the following theorem.

**Theorem 2** *For a scalar model if the potential is of the form  $V = \beta \exp(\mp 2\phi)$ , then the scale factors must follow a power law solution i.e.  $H = ht^{-1}$ .*

**Proof.** As above we perform the proof by using the Lie group method. In this case we must study the following ODE

$$\phi'' + \phi' H + \frac{d}{d\phi} V = 0,$$

where  $V = \beta \exp(\mp 2\phi)$ , then we shall study the different forms for the function  $H(t)$  in order to get and integrable ODE.

We have the next system of PDEs

$$\xi_{\phi\phi} = 0, \quad (59)$$

$$2H\xi_{\phi} + \eta_{\phi\phi} - 2\xi_{t\phi} = 0, \quad (60)$$

$$-6e^{-2\phi}\xi_{\phi} + H\xi_t - H'\xi + 2\eta_{t\phi} - \xi_{tt} = 0, \quad (61)$$

$$4e^{-2\phi}\eta + \eta_{tt} - 4e^{-2\phi}\xi_t + 2e^{-2\phi}\eta_{\phi} + H\eta_t = 0, \quad (62)$$

As we can easily see, the symmetry  $\xi = t, \eta = 1$ , brings us to get  $\phi = \ln t$ , as invariant solution, and from Eq. (61) we obtain the result i.e.  $H = ht^{-1}$ ,  $h \in \mathbb{R}$ . ■

Obviously this result does not mean that the solution must be self-similar, it only means that the scale factor must follow a power-law solution i.e. they are of the form  $a_i(t) = a_0 t^{a_j}$ , with  $a_j \in \mathbb{R}^+$ .

### 4.1 Scalar model

We write the FE in the following form:

$$\frac{a'}{a} \frac{b'}{b} + \frac{a'}{a} \frac{d'}{d} + \frac{d'}{d} \frac{b'}{b} - \left(2 + \frac{b^2}{d^2} + \frac{d^2}{b^2}\right) \frac{m^2}{4a^2} = 8\pi G\rho_{\phi}, \quad (63)$$

$$\frac{b''}{b} + \frac{d''}{d} + \frac{d'}{d} \frac{b'}{b} + \left(2 + \frac{b^2}{d^2} + \frac{d^2}{b^2}\right) \frac{m^2}{4a^2} = -8\pi Gp_{\phi}, \quad (64)$$

$$\frac{a''}{a} + \frac{b''}{b} + \frac{a'}{a} \frac{b'}{b} - \left(2 + \frac{3d^2}{b^2} - \frac{b^2}{d^2}\right) \frac{m^2}{4a^2} = -8\pi Gp_{\phi}, \quad (65)$$

$$\frac{b''}{b} - \frac{d''}{d} + \frac{a'}{a} \frac{b'}{b} - \frac{a'}{a} \frac{d'}{d} + m^2 \left(\frac{b^2}{d^2 a^2} - \frac{d^2}{b^2 a^2}\right) = 0, \quad (66)$$

$$\frac{d''}{d} + \frac{a''}{a} + \frac{a'}{a} \frac{d'}{d} - \left(2 + \frac{3b^2}{d^2} - \frac{d^2}{b^2}\right) \frac{m^2}{4a^2} = -8\pi Gp_{\phi}, \quad (67)$$

$$\phi'' + \phi'^2 H + \frac{d}{d\phi} V = 0. \quad (68)$$

By assuming the potential given by Eq. (57) it is possible to find the next set of solutions

$$a_{2\pm} = \frac{1}{2} \left(1 \pm \sqrt{1 - 4m^2}\right), \quad \forall m \in \left[-\frac{1}{2}, \frac{1}{2}\right] \setminus \{0\},$$

$$a(t) = a_0 (t + t_0),$$

$$b(t) = b_0 (t + t_0)^{a_{2\pm}},$$

$$d = d_0 (t + t_0)^{a_{2\pm}}, \quad (69)$$

and

$$\alpha_{\pm} = 1 \pm \sqrt{1 - 4m^2}, \quad \beta_{\pm} = \frac{1}{2} \left(1 \pm \sqrt{1 - 4m^2}\right)^2, \quad (70)$$

$$\phi = \pm \sqrt{\alpha_{\pm}} \ln(t + t_0), \quad V = \beta_{\pm} \exp\left(\mp \frac{2}{\sqrt{\alpha_{\pm}}} \phi\right). \quad (71)$$

As it is observed, we have obtained the same behavior for the scale factor as the one obtained in the case of a perfect fluid with time-varying constants model. For this reason, as we already know, we get:  $q > 0, \forall m \in [-\frac{1}{2}, \frac{1}{2}] \setminus \{0\}$ ,  $\mathcal{A} = const \in (0, 0.06) \rightarrow 0$ , while the Weyl parameter and the gravitational entropy behaves as  $\mathcal{W}^2 = const \in (0, 0.0016) \ll 1$ , and  $P^2 = const \in (-\infty, 0.01)$ ,  $\forall m \in [-\frac{1}{2}, \frac{1}{2}] \setminus \{0\}$ . Therefore the model does not accelerate but isotropizes since the quantities ( $\mathcal{A}$  and  $\mathcal{W}^2$ ) instead of being constant, they take values very close to zero. With regard to the quantity  $P^2$  it is observed that  $P^2 \rightarrow 0$ , (it takes values very close to zero)  $\forall m \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$  and it only runs to minus infinity when  $m \rightarrow \pm \frac{1}{2}$ .

### 4.2 Non-interacting scalar and matter fields

The stress-energy tensor may be written in the following form:  $T = T^m + T^{\phi}$ , where the energy density and the pressure of the fluid due a scalar field are given by Eq. (48). This describe a non-interacting dark matter and dark energy cosmological model (we assume that the baryon component can be ignored). Since the nature of both dark energy and dark matter is still unknown, there is no physical argument to exclude the possible non-interaction between them.

We write the FE in the following form:

$$\frac{a' b'}{a b} + \frac{a' d'}{a d} + \frac{d' b'}{d b} - \left(2 + \frac{b^2}{d^2} + \frac{d^2}{b^2}\right) \frac{m^2}{4a^2} = 8\pi G (\rho_m + p_\phi), \quad (72)$$

$$\frac{b''}{b} + \frac{d''}{d} + \frac{d' b'}{d b} + \left(2 + \frac{b^2}{d^2} + \frac{d^2}{b^2}\right) \frac{m^2}{4a^2} = -8\pi G (\omega\rho_m + p_\phi), \quad (73)$$

$$\frac{a''}{a} + \frac{b''}{b} + \frac{a' b'}{a b} - \left(2 + \frac{3d^2}{b^2} - \frac{b^2}{d^2}\right) \frac{m^2}{4a^2} = -8\pi G (\omega\rho_m + p_\phi), \quad (74)$$

$$\frac{b''}{b} - \frac{d''}{d} + \frac{a' b'}{a b} - \frac{a' d'}{a d} + m^2 \left(\frac{b^2}{d^2 a^2} - \frac{d^2}{b^2 a^2}\right) = 0, \quad (75)$$

$$\frac{d''}{d} + \frac{a''}{a} + \frac{a' d'}{a d} - \left(2 + \frac{3b^2}{d^2} - \frac{d^2}{b^2}\right) \frac{m^2}{4a^2} = -8\pi G (\omega\rho_m + p_\phi), \quad (76)$$

and the conservation equations now read

$$\rho'_m + (\omega + 1) \rho_m H = 0, \quad (77)$$

and

$$\phi'' + \phi' H + \frac{d}{d\phi} V = 0. \quad (78)$$

where  $H = h(t + t_0)^{-1}$ , and  $h = 1 + 2a_2$ .

In this case we have found the next solutions

$$\begin{aligned} a_2 &= \frac{1 - \omega}{2\omega + 2} \in (0, 1) \\ m &= \frac{1}{2} \frac{\sqrt{-(3\omega + 1)(\omega - 1)}}{1 + \omega} \in \left(0, \frac{1}{2}\right], \\ \beta &= \alpha a_2 = \alpha \left(\frac{1 - \omega}{2\omega + 2}\right) \\ \rho_0 &= \frac{1 - \omega - \alpha(1 + \omega)}{(1 + \omega)^2} > 0, \end{aligned} \quad (79)$$

where the constant  $\alpha$  must verify the condition:  $\alpha > \frac{1-\omega}{\omega+1} > 0$ ,  $\forall \omega \in (-\frac{1}{3}, 1)$ . Therefore we have the following behaviour for the main quantities

$$\rho_m = \rho_0 (t + t_0)^{-(1+\omega)h}, \quad p_m = \omega\rho_m, \quad (80)$$

$$a_1 = a_0 (t + t_0), \quad b = b_0 (t + t_0)^{a_2}, \quad d = d_0 (t + t_0)^{a_2}, \quad (81)$$

$$\phi = \pm\sqrt{\alpha} \ln(t + t_0), \quad V = \beta \exp\left(\mp \frac{2}{\sqrt{\alpha}} \phi\right). \quad (82)$$

Since the scale factor behaves as in the perfect fluid solution (see above) then the deceleration parameter (as we already know) behaves as:  $q = \frac{1}{2}(1 + 3\omega) > 0$ ,  $\mathcal{A} = \frac{(3\omega+1)^2}{16} = \text{const.} \in (0, 1)$ ,  $\forall \omega \in (-\frac{1}{3}, 1)$ . Therefore, with the above restrictions on the  $\omega$ -parameter our model does not inflate,  $q > 0$ . While the Weyl parameter and the

gravitational entropy behave as:  $\mathcal{W}^2 = \text{const} \in (0, 0.012)$ , and  $P^2 = \text{const.} \in (-\infty, a]$ , with  $a \rightarrow 0^+$  (see the above discussion about these quantities).

### 4.3 G-varying

We would like to study how the gravitational varies constant when we are considering only a scalar field. For this purpose, in analogy with the perfect fluid case (see [58]) and in a phenomenological way, by using the Bianchi identity  $\text{div}(8\pi G(t)T_{ij}) = 0$ , we propose the following conservation equation

$$G\rho' + G(\rho + p)H = -G'\rho \iff \phi' \left(\square\phi + \frac{dV}{d\phi}\right) = -\frac{G'}{G}\rho_\phi, \quad (83)$$

which is the modified KG equation.

$$\frac{a' b'}{a b} + \frac{a' d'}{a d} + \frac{d' b'}{d b} - \left(2 + \frac{b^2}{d^2} + \frac{d^2}{b^2}\right) \frac{m^2}{4a^2} = 8\pi G(t)\rho_\phi, \quad (84)$$

$$\frac{b''}{b} + \frac{d''}{d} + \frac{d' b'}{d b} + \left(2 + \frac{b^2}{d^2} + \frac{d^2}{b^2}\right) \frac{m^2}{4a^2} = -8\pi G(t)p_\phi, \quad (85)$$

$$\frac{a''}{a} + \frac{b''}{b} + \frac{a' b'}{a b} - \left(2 + \frac{3d^2}{b^2} - \frac{b^2}{d^2}\right) \frac{m^2}{4a^2} = -8\pi G(t)p_\phi, \quad (86)$$

$$\frac{b''}{b} - \frac{d''}{d} + \frac{a' b'}{a b} - \frac{a' d'}{a d} + m^2 \left(\frac{b^2}{d^2 a^2} - \frac{d^2}{b^2 a^2}\right) = 0, \quad (87)$$

$$\frac{d''}{d} + \frac{a''}{a} + \frac{a' d'}{a d} - \left(2 + \frac{3b^2}{d^2} - \frac{d^2}{b^2}\right) \frac{m^2}{4a^2} = -8\pi G(t)p_\phi, \quad (88)$$

$$\phi'' + H\phi' + \frac{dV}{d\phi} = -\frac{G'}{G} \frac{1}{\phi'} \rho_\phi. \quad (89)$$

In order to solve the FE we need to solve Eq. (83). To that end, we shall study it through the LG method. Eq. (83) could be rewritten in the following form

$$\phi''\phi' + ht^{-1}\phi'^2 + \frac{dV}{d\phi}\phi' + \rho_\phi \frac{G'}{G} = 0, \quad (90)$$

where  $H = ht^{-1}$ . For simplicity, and without lost

of generality, we consider  $H = ht^{-1}$  instead of its non-singular form. As above, we are seeking the forms of  $V(\phi)$  and  $G(t)$  for which our field equations admit symmetries and therefore they are integrable

As above, in order to study the possible solutions to Eq. (90) we apply the LG method, where the standard procedure brings us to get the following system of PDEs

$$\xi_{\phi\phi} = 0, \quad (91)$$

$$\eta_{\phi\phi} - 2\xi_{t\phi} + \left(2ht^{-1} + \frac{G'}{G}\right)\xi_\phi = 0, \quad (92)$$

$$2\eta_{t\phi} + \left(\frac{1}{2}\left(\frac{G''}{G} - \left(\frac{G'}{G}\right)^2\right) - ht^{-2}\right)\xi + 3V_\phi\xi_\phi + \left(ht^{-1} + \frac{G'}{2G}\right)\xi_t - \xi_{tt} = 0, \quad (93)$$

$$\eta_{tt} + V_{\phi\phi}\eta + 4\frac{G'}{G}V\xi_\phi + \left(ht^{-1} + \frac{G'}{2G}\right)\eta_t + 2V_\phi\xi_t - V_\phi\eta_\phi = 0, \quad (94)$$

$$\left(\frac{G''}{G} - \left(\frac{G'}{G}\right)^2\right)V\xi + \frac{G'}{G}V_\phi\eta + 3\frac{G'}{G}V\xi_t - 2\frac{G'}{G}V\eta_\phi = 0, \quad (95)$$

$$\frac{G'}{G}V\eta_t = 0, \quad (96)$$

where,  $V_\phi = \frac{dV}{d\phi}$ . The following symmetry

$$\xi = \frac{-t}{\alpha}, \quad \eta = \phi \implies \phi = t^{-\alpha} \quad (97)$$

then we obtain the following restrictions, from Eqs. (93-95). From Eq. (93) we get

$$G'' = \frac{G''^2}{G} - \frac{G'}{t} \implies G = \kappa_1 t^g, \quad (98)$$

while from Eq. (94) we obtain

$$V_{\phi\phi}\phi - \left(\frac{2}{\alpha} + 1\right)V_\phi = 0 \implies V = \kappa_2\phi^{2\left(\frac{1}{\alpha}+1\right)}, \quad (99)$$

in this way  $V = \kappa_2(t+t_0)^{-2(\alpha+1)}$ . So we have found that the main quantities behave as follows

$$\begin{aligned} \phi &= (t+t_0)^{-\alpha}, \\ V &= \kappa_2\phi^{2\left(\frac{1}{\alpha}+1\right)} = \kappa_2(t+t_0)^{-2(\alpha+1)}, \\ G &= \kappa_1(t+t_0)^g, \end{aligned} \quad (100)$$

such that  $g-2(\alpha+1) = -2$ , and therefore  $g = 2\alpha$ . Note that we may redefine the constants in order to get  $V \approx \phi^{-\alpha}$ .

**Theorem 3** *The only compatible form for the potential  $V(\phi)$  with the FE for a spacetime admitting a HFV, HO, where  $G = G(t)$ , is  $V(\phi) = V_0\phi^{-\alpha}$  and therefore  $\phi = (t+t_0)^\beta$ , and  $G = \kappa_1(t+t_0)^g$ , with  $\alpha, \beta, g \in \mathbb{R}$ .*

Taking into account all these results we have found the next solution

$$\begin{aligned} a_2 &= \frac{1}{2}\left(1 \pm \sqrt{1-4m^2}\right), \quad \forall m \in \left[-\frac{1}{2}, \frac{1}{2}\right] \setminus \{0\}, \\ \alpha &= \alpha > 0, \quad G_0 = \text{const.} > 0, \\ \beta &= \frac{(1-2m^2)}{G_0} + \sqrt{(1-4m^2)}, \end{aligned} \quad (101)$$

so the behaviour of the main quantities is the following one

$$a_1 = a_0(t+t_0), \quad b = b_0(t+t_0)^{a_2}, \quad d = d_0(t+t_0)^{a_2}, \quad (102)$$

$$\begin{aligned} \phi &= \phi_0(t+t_0)^{-\alpha}, \quad V = \beta(t+t_0)^{-2(\alpha+1)}, \\ G &= G_0(t+t_0)^{2\alpha}. \end{aligned} \quad (103)$$

Notice that this is the same solution for the scale factor than in the above models. Therefore, with

the above restrictions on the  $m$ -parameter our model does not inflate  $q > 0$ . i.e. the model does not accelerate but isotropizes since  $\mathcal{W}^2 \rightarrow 0$  as well as  $\mathcal{A}$ . With regard to the quantity  $P^2$  it is observed that  $P^2 \rightarrow 0, \forall m \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$  and it only runs to minus infinity when  $m \rightarrow \pm\frac{1}{2}$ . With regard to the gravitational constant  $G$ , we have obtained that it is an increasing time function since  $\alpha > 0$ .

#### 4.4 $G$ variable with matter and a scalar field

We start by rewriting the stress-energy tensor in the following way,  $T^{ij} = T_m^{ij} + T_\phi^{ij}$ , where  $T^{ij} = (\tilde{p} + \tilde{\rho})u^i u^j + \tilde{p}g^{ij}$ , and  $\tilde{\rho} = \rho_m + \rho_\phi$  and  $\tilde{p} = p_m + p_\phi$ , and taking into account the Bianchi identity

$$\begin{aligned} \text{div}(8\pi G(t)T_{ij}) &= 0 \\ G\tilde{\rho}' + G(\tilde{p} + \tilde{\rho})H &= -G'\tilde{\rho}, \end{aligned} \quad (104)$$

$$\frac{a'}{a} \frac{b'}{b} + \frac{a'}{a} \frac{d'}{d} + \frac{d'}{d} \frac{b'}{b} - \left(2 + \frac{b^2}{d^2} + \frac{d^2}{b^2}\right) \frac{m^2}{4a^2} = 8\pi G(t)(\rho_m + \rho_\phi), \quad (108)$$

$$\frac{b''}{b} + \frac{d''}{d} + \frac{d'}{d} \frac{b'}{b} + \left(2 + \frac{b^2}{d^2} + \frac{d^2}{b^2}\right) \frac{m^2}{4a^2} = -8\pi G(t)(\omega\rho_m + p_\phi), \quad (109)$$

$$\frac{a''}{a} + \frac{b''}{b} + \frac{a'}{a} \frac{b'}{b} - \left(2 + \frac{3d^2}{b^2} - \frac{b^2}{d^2}\right) \frac{m^2}{4a^2} = -8\pi G(t)(\omega\rho_m + p_\phi), \quad (110)$$

$$\frac{b''}{b} - \frac{d''}{d} + \frac{a'}{a} \frac{b'}{b} - \frac{a'}{a} \frac{d'}{d} + m^2 \left(\frac{b^2}{d^2 a^2} - \frac{d^2}{b^2 a^2}\right) = 0, \quad (111)$$

$$\frac{d''}{d} + \frac{a''}{a} + \frac{a'}{a} \frac{d'}{d} - \left(2 + \frac{3b^2}{d^2} - \frac{d^2}{b^2}\right) \frac{m^2}{4a^2} = -8\pi G(t)(\omega\rho_m + p_\phi), \quad (112)$$

$$\rho'_m + (\rho_m + p_m)H = -\frac{G'}{G}\rho_m, \quad (113)$$

$$\phi'' + H\phi' + \frac{dV}{d\phi} = -\frac{G'}{G} \frac{1}{\phi'} \left(\frac{1}{2}\phi'^2 + V(\phi)\right). \quad (114)$$

With a potential given by Eq. (100) we have found the next solution

$$\begin{aligned} a_2 &= \frac{1}{2} \left(1 \pm \sqrt{1 - 4m^2}\right) \quad \forall m \in \left[-\frac{1}{2}, \frac{1}{2}\right] \setminus \{0\}, \quad \alpha = \alpha > 0, \\ \beta &= \frac{3\omega \left(1 \pm \sqrt{(1 - 4m^2)}\right) + G_0\alpha(1 - \omega) + 1 \pm \sqrt{(1 - 4m^2)} - 4m^2(1 + \omega)}{2G_0(\omega + 1)}, \\ \rho_0 &= \frac{1 \pm \sqrt{(1 - 4m^2)} - G_0\alpha^2}{G_0(\omega + 1)}, \quad G_0 = \text{const.} > 0, \end{aligned} \quad (115)$$

thus

$$\begin{aligned} \rho_m &= \rho_0 (t + t_0)^{-((\omega+1)h+2\alpha)}, \\ p_m &= \omega\rho_m, \end{aligned} \quad (116)$$

$$\begin{aligned} a_1 &= a_0 (t + t_0), \quad b = b_0 (t + t_0)^{a_2}, \\ d &= d_0 (t + t_0)^{a_2}, \end{aligned} \quad (117)$$

i.e.

$$\rho'_m + (\rho_m + p_m)H + \phi' \left(\square\phi + \frac{dV}{d\phi}\right) = -\frac{G'}{G}(\rho_m + p_\phi). \quad (105)$$

We may study Eq. (105) in several ways. One of them, maybe the simplest one, may be splitting it into

$$\rho'_m + (\rho_m + p_m)H = -\frac{G'}{G}\rho_m, \quad (106)$$

$$\phi' \left(\square\phi + \frac{dV}{d\phi}\right) = -\frac{G'}{G}\rho_\phi. \quad (107)$$

Notice that this approach is similar to a scenario describing an interacting scalar and matter fields.

There the FE for this model are:

$$\begin{aligned} \phi &= \phi_0 (t + t_0)^{-\alpha}, \quad V = \beta (t + t_0)^{-2(\alpha+1)}, \\ G &= G_0 (t + t_0)^{2\alpha}. \end{aligned} \quad (118)$$

Therefore, as in the last model, with the above restrictions on the  $m$ -parameter our model does not accelerate but isotropizes since  $\mathcal{W}^2 \rightarrow 0$  as well as  $\mathcal{A}$ . With regard to the quantity  $P^2$  it is observed that  $P^2 \rightarrow 0, \forall m \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$  and only runs to minus infinity when  $m \rightarrow \pm \frac{1}{2}$ .  $G$  is an increasing time function as in the above model.

## 5 Scalar-tensor model

We consider the following field equations for the BD model [59],

$$\begin{aligned} R_{ij} - \frac{1}{2}g_{ij}R &= \frac{8\pi}{\phi}T_{ij}^m + \Lambda(\phi)g_{ij} + \frac{\omega}{\phi^2}(\phi_{,i}\phi_{,j} \\ &\quad - \frac{1}{2}g_{ij}\phi_{,l}\phi^{,l}) + \frac{1}{\phi}(\phi_{;ij} - g_{ij}\square\phi), \end{aligned} \quad (119)$$

$$\begin{aligned} \square\phi + \frac{1}{2}\phi_{,l}\phi^{,l}\frac{d}{d\phi}\ln\left(\frac{\omega(\phi)}{\phi}\right) + \frac{1}{2}\frac{\phi}{\omega(\phi)}(R + 2 \\ \times \frac{d}{d\phi}(\phi\Lambda(\phi))) = 0. \end{aligned} \quad (120)$$

The arbitrary functions  $\omega(\phi)$  and  $\Lambda(\phi)$  distinguish the different scalar-tensor theories of gravitation.  $\Lambda(\phi)$  is a potential function and plays the role of a cosmological constant, and  $\omega(\phi)$  is the coupling function of the particular theory.  $T_{ij}^m$  is the matter stress-energy tensor.

The last equation can be substituted by

$$\begin{aligned} \square\phi + \frac{2}{3 + 2\omega(\phi)}\left(\phi^2\frac{d\Lambda}{d\phi} - \phi\Lambda(\phi)\right) &= \frac{1}{(3 + 2\omega(\phi))} \\ &\times\left(8\pi T - \frac{d\omega}{d\phi}\phi_{,l}\phi^{,l}\right), \end{aligned} \quad (121)$$

where  $T = T_i^i$  is the trace of the stress-energy tensor, where we have assumed  $\phi = \phi(t)$ , and the derivatives respect  $t$  are denoted by a comma. Furthermore it is verified the following relationship:  $divT = 0, i.e.$

$$\rho' + (\rho + p)H = 0. \quad (122)$$

In what follows we shall assume  $\omega(\phi) = const, \Lambda = \Lambda(\phi)$ . The corresponding field equations with a perfect fluid for the matter content in the homogeneous line element (Bianchi VI<sub>0</sub> model) will be calculated. Thus the field equations are

$$\frac{a'}{a} \frac{b'}{b} + \frac{a'}{a} \frac{d'}{d} + \frac{d'}{d} \frac{b'}{b} - \left(2 + \frac{b^2}{d^2} + \frac{d^2}{b^2}\right) \frac{m^2}{4a^2} = \frac{8\pi}{\phi} \rho - H \frac{\phi'}{\phi} + \frac{\omega}{2} \left(\frac{\phi'}{\phi}\right)^2 + \Lambda(\phi), \quad (123)$$

$$\frac{b''}{b} + \frac{d''}{d} + \frac{d'}{d} \frac{b'}{b} + \left(2 + \frac{b^2}{d^2} + \frac{d^2}{b^2}\right) \frac{m^2}{4a^2} = -\frac{8\pi}{\phi} p - \frac{\phi'}{\phi} \left(\frac{d'}{d} + \frac{b'}{b}\right) - \frac{\omega}{2} \left(\frac{\phi'}{\phi}\right)^2 - \frac{\phi''}{\phi} + \Lambda(\phi), \quad (124)$$

$$\begin{aligned} \frac{a''}{a} + \frac{b''}{b} + \frac{a'}{a} \frac{b'}{b} - \left(2 + \frac{3d^2}{b^2} - \frac{b^2}{d^2}\right) \frac{m^2}{4a^2} &= -\frac{8\pi}{\phi} p - \frac{\phi'}{\phi} \left(\frac{a'}{a} + \frac{b'}{b}\right) - \frac{\omega}{2} \left(\frac{\phi'}{\phi}\right)^2 - \frac{\phi''}{\phi} + \Lambda(\phi) \\ &\quad - \cosh(mx)^2 \frac{\phi'}{\phi} \left(\frac{d'}{d} - \frac{b'}{b}\right), \end{aligned} \quad (125)$$

$$\frac{b''}{b} - \frac{d''}{d} + \frac{a'}{a} \left(\frac{b'}{b} - \frac{d'}{d}\right) + \frac{m^2}{a^2} \left(\frac{b^2}{d^2} - \frac{d^2}{b^2}\right) = \frac{\phi'}{\phi} \left(\frac{b'}{b} - \frac{d'}{d}\right), \quad (126)$$

$$\begin{aligned} \frac{d''}{d} + \frac{a''}{a} + \frac{a'}{a} \frac{d'}{d} - \left(2 + \frac{3b^2}{d^2} - \frac{d^2}{b^2}\right) \frac{m^2}{4a^2} &= -\frac{8\pi}{\phi} p - \frac{\phi'}{\phi} \left(\frac{a'}{a} + \frac{d'}{d}\right) - \frac{\omega}{2} \left(\frac{\phi'}{\phi}\right)^2 - \frac{\phi''}{\phi} + \Lambda(\phi) \\ &\quad - \cosh(mx)^2 \frac{\phi'}{\phi} \left(\frac{d'}{d} - \frac{b'}{b}\right), \end{aligned} \quad (127)$$

$$(3 + 2\omega(\phi)) \left(\frac{\phi''}{\phi} + H \frac{\phi'}{\phi}\right) - 2 \left(\Lambda - \phi \frac{d\Lambda}{d\phi}\right) = \frac{8\pi}{\phi} (\rho - 3p), \quad (128)$$

$$\rho' + (\rho + p)H = 0. \quad (129)$$

Since we are only interested in finding self-similar solutions then, if we take into account our previous results, i.e.  $b = d$ , the FE reads

$$2\frac{a'b'}{ab} + \left(\frac{b'}{b}\right)^2 - \frac{m^2}{a^2} = \frac{8\pi}{\phi}\rho - H\frac{\phi'}{\phi} + \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 + \Lambda(\phi), \quad (130)$$

$$2\frac{b''}{b} + \left(\frac{b'}{b}\right)^2 + \frac{m^2}{a^2} = -\frac{8\pi}{\phi}p - 2\frac{\phi'b'}{\phi b} - \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 - \frac{\phi''}{\phi} + \Lambda(\phi), \quad (131)$$

$$\frac{a''}{a} + \frac{b''}{b} + \frac{a'b'}{ab} - \frac{m^2}{a^2} = -\frac{8\pi}{\phi}p - \frac{\phi'}{\phi}\left(\frac{a'}{a} + \frac{b'}{b}\right) - \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 - \frac{\phi''}{\phi} + \Lambda(\phi), \quad (132)$$

$$\frac{d''}{d} + \frac{a''}{a} + \frac{a'd'}{ad} - \frac{m^2}{a^2} = -\frac{8\pi}{\phi}p - \frac{\phi'}{\phi}\left(\frac{a'}{a} + \frac{b'}{b}\right) - \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 - \frac{\phi''}{\phi} + \Lambda(\phi), \quad (133)$$

and the conservation equations

$$(3 + 2\omega(\phi))\left(\frac{\phi''}{\phi} + H\frac{\phi'}{\phi}\right) - 2\left(\Lambda - \phi\frac{d\Lambda}{d\phi}\right) = \frac{8\pi}{\phi}(\rho - 3p), \quad (134)$$

$$\rho' + (\rho + p)H = 0, \Leftrightarrow \rho = \rho_0 t^{-\alpha}, \quad (135)$$

where  $H = h(t + t_0)^{-1}$ , with  $h = (1 + 2a_2)$ , and  $\alpha = h(1 + \gamma)$ , we are taking into account the equation of state  $p = \gamma\rho$ ,  $\gamma \in (-1, 1]$ .

In order to solve the resulting FE we need to integrate

$$(3 + 2\omega)\left(\frac{\phi''}{\phi} + \frac{h}{t}\frac{\phi'}{\phi}\right) - 2\left(\Lambda - \phi\frac{d\Lambda}{d\phi}\right) = \frac{8\pi(1 - 3\gamma)\rho_0}{\phi t^\alpha}, \quad (136)$$

We may study this equation through the LG method, i.e. we study the kind of functions  $\Lambda(\phi)$  such that this equation is integrable in a closed form. We start by rewriting it in an appropriate way

$$\phi'' + ht^{-1}\phi' - B\left(\Lambda - \phi\frac{d\Lambda}{d\phi}\right)\phi - Ct^{-\alpha} = 0, \quad (137)$$

where

$$h = (1 + 2a_2), \quad B = \frac{2}{(3 + 2\omega)}, \quad (138)$$

$$C = \frac{8\pi(1 - 3\gamma)\rho_0}{(3 + 2\omega)},$$

and we shall deduce that  $\alpha = 2 - n$ .

$$t^2\xi_{\phi\phi} = 0, \quad (139)$$

$$2ht^{-1}\xi_\phi + \eta_{\phi\phi} - 2\xi_{\phi t} = 0, \quad (140)$$

$$-3(B\phi(\Lambda - \phi\Lambda_\phi) + Ct^{-\alpha})\xi_\phi + ht^{-2}(t\xi_t - \xi) + 2\eta_{t\phi} - \xi_{tt} = 0, \quad (141)$$

$$B\eta(\phi^2\Lambda_{\phi\phi} - (\Lambda - \phi\Lambda_\phi)) - 2B\phi(\Lambda - \phi\Lambda_\phi)\xi_t + \frac{C}{t^{-\alpha}}\left(\alpha\frac{\xi}{t} - 2\xi_t\right) + \left(B\phi(\Lambda - \phi\Lambda_\phi) + C\frac{C}{t^{-\alpha}}\right)\eta_\phi + \frac{h}{t}\eta_t + \eta_{tt} = 0. \quad (142)$$

For example, if we impose the symmetry

$$\xi = t, \quad \eta = n\phi, \quad (143)$$

brings us to obtain the following restriction on  $\Lambda(\phi)$ . From Eq. (142) we get

$$B\phi(n\phi^2\Lambda_{\phi\phi} - 2(\Lambda - \phi\Lambda_\phi)) + Ct^{-\alpha}(\alpha - 2 + n) = 0, \quad (144)$$

and therefore, we find as result, that:

$$n = 2 - \alpha, \quad (145)$$

and

$$n\phi^2\Lambda_{\phi\phi} - 2(\Lambda - \phi\Lambda_\phi) = 0 \implies \Lambda = \phi^{-2/n}, \quad (146)$$

is a solution. In fact the most general solution is  $\Lambda = C_1\phi + C_2\phi^{-2/n}$ . This result is valid for all the self-similar Bianchi models. Furthermore, the symmetry Eq. (143) brings us to obtain a particular solution of Eq. (137) which is given by,  $\phi = \phi_0 t^n$ , with  $\phi_0 \in \mathbb{R}$ , with  $n = 2 - \alpha$ , and the following constrains on the numerical constants must be verified:  $n(n - 1) + hn - 2(1 + \frac{2}{n})B - C = 0$ ,  $\alpha = h(1 + \gamma) = (1 + 2a_2)(1 + \gamma)$ . We would like to emphasize that other solutions could be obtained with this procedure by imposing other symmetries.

We find the following solution:

$$\begin{aligned}
 \phi_0 &= \phi_0, \quad \phi_0 = 1, \quad \Lambda_0 = 0, \\
 a_2 &= -\frac{(\gamma - 1)(\omega(\gamma - 1) - 1)}{2\omega(\gamma^2 - 1) + \gamma - 3}, \quad a_2 = 0, \Leftrightarrow \gamma = 1, \\
 q &= \frac{2(\omega((3\gamma + 1)(\gamma - 1)) - 2)}{4\omega(\gamma - 1) + 3\gamma - 5}, \quad q = 0, \Leftrightarrow \gamma = A_{\pm}, \\
 \rho_0 &= -\frac{(3 + 2\omega)^2(\gamma - 1)^3}{8\pi(2\omega(\gamma^2 - 1) + \gamma - 3)}, \quad \rho_0 = 0, \Leftrightarrow \gamma = 1, \\
 m &= \frac{(\gamma - 1)\sqrt{-2(3 + 2\omega)(\omega((3\gamma + 1)(\gamma - 1)) - 2)}}{2(2\omega(\gamma^2 - 1) + \gamma - 3)}, \\
 m &= 0, \Leftrightarrow \gamma = 1 \wedge A_{\pm}, \\
 n &= \frac{-(3\gamma - 1)(\gamma - 1)}{2\omega(\gamma^2 - 1) + \gamma - 3}, \quad n = 0, \Leftrightarrow \gamma = 1 \wedge \frac{1}{3},
 \end{aligned} \tag{147}$$

where  $A_{\pm} = \frac{(\omega \pm \sqrt{2\omega(2\omega + 3)})}{3\omega}$ .

We get for the BD parameter  $\omega$  that according to solar system experiments is  $\omega \approx 500$ . (see [60]). A better estimation of this parameter should be obtained from measure of other cosmological parameters in order to constrain  $\omega$  more strongly than by means of solar system experiments (see [61]). However, theories of the very early Universe such as string theory, are better described in the context of JBD, which shows that  $\omega$  can take negative values (see for example [62]). A recent value for  $\omega$  is  $\omega \approx 3300$  [16].

If we fix  $\omega = 3300$ , then we get the following results

$$\begin{aligned}
 A_+ &= 1.0002, \quad A_- = -0.33348, \quad m > 0, \quad \forall \gamma \in (A_-, 1], \\
 a_2 \geq 0, \quad \forall \gamma \in (-1, 1], \quad q &= \begin{cases} < 0 & \forall \gamma < A_- \\ = 0 & \gamma = A_- \\ > 0 & \forall \gamma > A_- \end{cases}, \\
 \rho_0 \geq 0, \quad \forall \gamma \in (-1, 1], \quad n &= \begin{cases} > 0 & \forall \gamma < 1/3 \\ = 0 & \gamma = 1/3 \wedge 1 \\ < 0 & \forall \gamma \in (\frac{1}{3}, 1) \end{cases}.
 \end{aligned} \tag{148}$$

Therefore this solution, with  $\omega = 3300$ , is only valid  $\forall \gamma \in (A_-, 1]$ . Note that if  $\gamma < A_-$  then  $m$  is not defined. This means that

$$\begin{aligned}
 a &= a_0(t + t_0), \quad b = b_0(t + t_0)^{a_2}, \quad d = d_0(t + t_0)^{a_2}, \\
 H &= h(t + t_0)^{-1}, \quad \rho = \rho_0(t + t_0)^{-\alpha}, \quad \Lambda = 0, \\
 \phi &= \phi_0(t + t_0)^n
 \end{aligned} \tag{149}$$

and therefore the scale factors are increasing time functions, the energy density is a positive time

decreasing function. The solution does not inflate since  $q > 0 \quad \forall \gamma \in (A_-, 1]$ .  $\phi$  is a positive growing time function if  $\gamma \in (A_-, 1/3)$ , it is constant if  $\gamma = 1/3$  and it behaves as a decreasing time function if  $\gamma \in (1/3, 1)$ , if  $\gamma = 1$  then it behaves as a constant. This means that  $G$  is a decreasing time function if  $\gamma \in (A_-, 1/3)$ , it behaves as a true constant if  $\gamma = 1/3 \wedge 1$  and if  $\gamma \in (1/3, 1)$  then it is a growing time function. The cosmological ‘‘constant’’,  $\Lambda$  vanishes. Notice that  $\alpha = \frac{(4\omega(\gamma - 1) + 3\gamma - 5)(\gamma + 1)}{2\omega(\gamma^2 - 1) + \gamma - 3}$ .

In order to check if the solution isotropize we compute the quantities  $\mathcal{A}$  and  $\mathcal{W}^2$ , obtaining

$$\mathcal{A} = \frac{(\omega(\gamma^2 - 2\gamma - 1) - 2)^2}{(4\omega(\gamma - 1) + 3\gamma - 5)^2} = const \in (0, 1) \tag{150}$$

$$\begin{aligned}
 \mathcal{W}^2 &= -\frac{(\gamma - 1)^2(\omega(\gamma^2 - 2\gamma - 1) - 2)^2}{36(4\omega(\gamma - 1) + 3\gamma - 5)^4} \\
 &\quad \times (8\omega((2\gamma + 1)(\gamma - 1)) - 3\gamma^2 + 6\gamma - 15),
 \end{aligned} \tag{151}$$

$\mathcal{W}^2 = const \in (0, 0.01)$ ,  $\forall \gamma \in (A_-, 1]$  and  $\omega = 3300$ .  $\mathcal{A}(A_-) = \mathcal{W}^2(A_-) = 0$ . With regard to the gravitational entropy we have obtained the following behaviour:  $P^2(A_-) = 0$ , if  $\gamma \in (A_-, -0.2000969530)$  then  $P^2 > 0$ ,  $P^2(-0.2000969530) = 0$  and it runs to  $-\infty \quad \forall \gamma \in (-0.2000969530, 1]$ , so once again, we have checked that this quantity is not a good definition for gravitational entropy.

### 5.1 The particular case $\gamma = 1/3$ .

In this case  $T = 0$  and therefore we find the next solution

$$\begin{aligned}
 \phi_0 &= \phi_0, \quad \phi_0 = 1, \\
 \Lambda_0 &= \frac{1}{6} \left[ (3 + 2\omega) \left( 4m + \sqrt{3} \right)^2 \right], \\
 \Lambda_0 = 0 &\iff m = -\frac{\sqrt{3}}{4}, \\
 a_2 &= \frac{\sqrt{3}}{3} \left( 3m + \sqrt{3} \right), \quad a_2 = 0 \iff m = -\frac{\sqrt{3}}{3}, \\
 q &= \frac{-2m}{2m + \sqrt{3}}, \quad q = \begin{cases} > 0 & \forall m < 0 \\ < 0 & \forall m > 0 \end{cases},
 \end{aligned}$$

$$\begin{aligned} \rho_0 &= -\frac{1}{16\pi} \left( 32\omega \left( m + \frac{1}{4}\sqrt{3} \right)^2 + 9 + 24m\sqrt{3} + 44m^2 \right), \\ \alpha &= \frac{4\sqrt{3}}{3} \left( 2m + \sqrt{3} \right), \\ \rho_0 = 0 &\Leftrightarrow m = -\frac{\sqrt{3} (4\omega \pm \sqrt{2\omega + 3} + 6)}{16\omega + 22}, \\ n = -\frac{2\sqrt{3}}{3} \left( 4m + \sqrt{3} \right), \quad n = 0 &\Leftrightarrow m = -\frac{\sqrt{3}}{4}. \end{aligned} \tag{152}$$

If we fix  $\omega = 3300$ , then we get the following results

$$\begin{aligned} \rho_0 &> 0, \quad \forall m \in I, \quad \alpha > 0, \quad \forall m \in I, \\ \Lambda_0 \geq 0, \quad \forall m \in I, \quad \Lambda_0 = 0 &\Leftrightarrow m_{\Lambda_0} = -\frac{\sqrt{3}}{4} = -0.43301, \\ a_2 > 0, \quad \forall m \in I, \quad a_2 = 0 &\Leftrightarrow m = -\frac{\sqrt{3}}{3} \notin I, \\ q &> 0, \quad \forall m \in I, \\ n &= \begin{cases} > 0 & \forall m \in (-0.43569, m_{\Lambda_0}) \\ = 0 & m = m_{\Lambda_0} \\ < 0 & \forall m \in (m_{\Lambda_0}, -0.43036) \end{cases}, \end{aligned} \tag{153}$$

therefore this solution, with  $\omega = 3300$ , is only valid  $\forall m \in I$ , where  $I = (-0.43569, -0.43036)$ . Note that  $m_{\Lambda_0} \in I$ ,

$$\begin{aligned} \Lambda_0(m_{\Lambda_0}) = 0, \quad a_2(m_{\Lambda_0}) = \frac{1}{4}, \quad q(m_{\Lambda_0}) = 1, \\ \rho_0(m_{\Lambda_0}) = 1.4921 \times 10^{-2}. \end{aligned} \tag{154}$$

This means that

$$\begin{aligned} a = a_0(t + t_0), \quad b = b_0(t + t_0)^{a_2}, \quad d = d_0(t + t_0)^{a_2}, \\ H = h(t + t_0)^{-1}, \quad \rho = \rho_0(t + t_0)^{-\alpha}, \quad \phi = \phi_0(t + t_0)^n \end{aligned} \tag{155}$$

and therefore the scale factors are growing time functions, the energy density is a positive time decreasing function. The solution does not inflate since  $q > 0 \forall m \in I$ .  $\phi$  is a positive growing time function if  $m \in (-0.43569, m_{\Lambda_0})$ , it is constant if  $m = m_{\Lambda_0}$  and it behaves as a decreasing time function if  $m \in (m_{\Lambda_0}, -0.43036)$ . This means that  $G$  is a decreasing time function if  $m \in (-0.43569, m_{\Lambda_0})$ , it behaves as a true constant if  $m = m_{\Lambda_0}$  and if  $m \in (-0.43569, m_{\Lambda_0})$  then it is a growing time function. The cosmological “constant”,  $\Lambda$  is a positive decreasing time function except in  $m = m_{\Lambda_0}$ .

## 6 Conclusions

In this paper we have studied some Bianchi types VI<sub>0</sub> (with an unusual metric) models under the self-similarity hypothesis. We have started by comparing our results with the “classical” perfect fluid solution already studied by Collins, Wainwright and Hsu and other authors [24]. Furthermore, we have been able to improve the solutions since we have found a non-singular solution for the scale factors i.e. they behave as  $a(t) \sim (t + t_0)^{a_1}$ . However, the metric employed in this paper is very restrictive, since it allows us to obtain less solutions than with the usual one [24]. Nevertheless we have been able to obtain a new solution for the case of a perfect fluid with time-varying constants. This solution is not inflationary but it is very close to isotropizing since the quantities  $\mathcal{A}$  and  $\mathcal{W}^2$  take values very close to zero. In fact, for an adequate selection of the parameters  $\omega$  and  $m$  they run to zero. This solution is valid for all  $\omega \in (-1, 1]$  and  $m \in [-1/2, 1/2]$ . In this case we have been able to enlarge the range of validity for the equation of state and we have shown that if  $G$  behaves as a growing time function then  $\Lambda$  is a “positive” decreasing time function. In the same way, if  $G$  is decreasing then  $\Lambda$  behaves as a “negative” decreasing time function. With regard to the gravitational entropy, we have come to the conclusion that the quantity  $P^2$  is not an acceptable candidate for gravitational entropy along the homothetic trajectories of any self-similar spacetime (in all the cases studied in this paper).

In the second model we have studied a scalar field. We have started this section by calculating the potentials compatible with the self-similar solutions. Inversely, we have proved that for such scalar fields the scale factor must follow a power law solution. These theorems are very general and are valid for all Bianchi models. We have studied two cases. From the first one, with a scalar field alone, we have obtained a solution that is not inflationary but it could be considered to be very close to isotropize, since, as above, since the quantities  $\mathcal{A}$  and  $\mathcal{W}^2$  take values very close to zero. In the second case, we have studied a non-interacting scalar and matter fields. The solution is not inflationary but isotropize as in the previous cases, and it is valid  $\forall \omega \in (-1/3, 1)$  and  $m \in (0, 1/2]$ . In order to incorporate into this framework a variable  $G$ , we have proposed, in a phenomenological way, a



modified Klein-Gordon equation. We have studied the kind of potential compatible with a self-similar solution and a variable  $G$ . Once we have deduced the potential and the scalar field then we study two cases, a scalar field with a  $G$ -var and a scalar field with a matter field. The conservation equation outlined in this case is quite similar to the one employed in the case of interacting scalar fields. The solutions obtained are similar since the scale factor is the same and therefore they are not inflationists and close to isotropize. In both cases,  $G$  behaves as a positive increasing time function.

In the scalar-tensor model, for simplicity, we have chosen,  $\omega(\phi) = \text{const.}$  and  $\Lambda(\phi)$  playing the role of an effective cosmological constant. As we have shown, the resulting FE are quite difficult to study. Nevertheless, since we are only interested in studying self-similar solutions, we have been able to simplify the FE. We would like to stress that we have not needed to make any assumption in order to integrate them. By using the Lie group method we have obtained a possible form for the dynamical cosmological constant,  $\Lambda(\phi) = \phi^{-2/n} = t^{-2}$ , since  $\phi = \phi_0 t^m$ . In the same way, we emphasize that this result is valid for all the self-similar Bianchi models. With this result we have obtained several solutions for the model. We have considered that the first of the obtained solutions is unphysical since  $\rho_0 < 0$  if the coupling parameter  $\omega$  is positive as the recent observations suggest. Nevertheless if we consider  $\omega < 0$ , as it is suggested by the string theories [62], this solution has physical meaning. In the second solution,  $\Lambda = 0$  and it is only valid when  $\gamma \in I$ , where  $I = (A_-, 1]$ , where  $A_- = -0.33348$  if  $\omega = 3300$  as recent experiments suggest [16]. For such values  $\phi$  behaves as a growing time function if  $\gamma \in (A_-, 1/3)$ , it is constant if  $\gamma = 1/3 \wedge 1$  and it is a positive decreasing time function in the interval  $\gamma \in (1/3, 1)$ . Therefore,  $G$  is decreasing, constant and growing in the same intervals. In the same way we have found that this solution does not inflate since  $q > 0$ ,  $\gamma \in I$ , which is unusual. We may also say that this solution would be considered to be very close to an isotropy state since the Weyl parameter,  $\mathcal{W}^2 \ll 1$ . In fact this quantity takes values very close to zero  $\gamma \in I$ . We have also studied the particular solution  $\gamma = 1/3$ . In this case the trace of the stress-energy tensor vanishes and therefore the conservation equation reduces to a very simple ODE. This solution is only valid for a very

restrictive interval,  $m \in I_m = (m_1, m_2)$ , where  $\rho_0 > 0$ . We have found that in this case  $\Lambda$  behaves as a positive decreasing time function except when  $m = m_{\Lambda_0} \in I$ , for which,  $\Lambda(m_{\Lambda_0}) = 0$ . In the same way, we have found that  $\phi$  is a growing time function when  $m \in (m_1, m_{\Lambda_0})$ , it is constant if  $m = m_{\Lambda_0}$  and it behaves as a decreasing time function if  $m \in (m_{\Lambda_0}, m_2)$ .  $G$  behaves in the inverse way in the same intervals. This solution does not inflate as the one above does. Finally we would like to stress the fact that, in this solution, the exponents,  $a_2$ ,  $n$  and  $\alpha$  are irrational numbers. This could have implications for the integrability of the FE (see [63, 64]).

## Appendix

### A Matter collineation for the scalar model

In this section we shall study the matter collineations for the scalar field following a method employed in [54]. We start by defining a generic vector field  $X = (X_i(t, x, y, z))_{i=1}^4 \in \mathfrak{X}(M)$ . The energy-momentum tensor is defined by Eq. (22). The metric tensor  $g_{ij}$  is defined by Eq. (6). In recent years, much interest has been shown in the study of matter collineation (MCs) (see for example [65]-[73]). A vector field along which the Lie derivative of the energy-momentum tensor vanishes is called an MC, i.e.  $\mathcal{L}_X T_{ij} = 0$ , where  $X^i$  is the symmetry or collineation vector. Also, assuming the Einstein field equations, a vector  $X^i$  generates an MC if  $\mathcal{L}_X G_{ij} = 0$ . It is obvious that the symmetries of the metric tensor (isometries) are also symmetries of the Einstein tensor  $G_{ij}$ , but this is not necessarily the case for the symmetries of the Ricci tensor (Ricci collineations) which are not, in general, symmetries of the Einstein tensor. If  $X$  is a Killing vector (KV) (or a homothetic vector), then  $\mathcal{L}_X T_{ij} = 0$ , thus every isometry is also an MC but the converse is not true, in general. Notice that collineations can be proper (non-trivial) or improper (trivial). A proper MC is defined to be an MC which is not a KV, or a homothetic vector. Carot et al (see [66]) and Hall et al. (see [67]) have noticed some important general results about the Lie algebra of MCs. Let  $M$  be a spacetime manifold. Then, generically, any vector field  $X \in \mathfrak{X}(M)$  which simultaneously satisfies

$\mathcal{L}_X T_{ab} = 0$  ( $\Leftrightarrow \mathcal{L}_X G_{ab} = 0$ ) and  $\mathcal{L}_X C_{bcd}^a = 0$  is a homothetic vector field i.e.  $\mathcal{L}_X g = 2g$ .

The usual matter collineation equations read,  $\mathcal{L}_X T_{ij}^\phi = 0$ ,  $T_{ij}^\phi$  is given by Eq. (48), finding, in this case, that the obtained matter collineation (MC) is:

$$X = X_1 \partial_t + (X'_1 - X_1 H_2) y \partial_y + (X'_1 - X_1 H_2) z \partial_z, \quad (\text{A.1})$$

which is a proper MC, and where, as it is observed, if  $X_1 = (t + t_0)$ , then it is regained the usual homothetic vector field (see Eq. (13)) i.e. a improper MC.

For this reason we may also check that the homothetic vector field also verifies the equation,  $\mathcal{L}_{HO} T_{ij} = 0$ , (note that it is verified  $\mathcal{L}_{HO} C_{bcd}^a = 0$ ) that we may develop as follows:

$$\begin{aligned} \rho' t + 2\rho &= 0, \\ aa' + taa'' - t(a')^2 &= 0, \\ g_{33}y \left( \frac{b'}{b} + t \frac{b''}{b} - t \frac{b'^2}{b^2} \right) + g_{34}z \left( \frac{d'}{d} + t \frac{d''}{d} - t \frac{d'^2}{d^2} \right) &= 0, \\ g_{34}y \left( \frac{b'}{b} + t \frac{b''}{b} - t \frac{b'^2}{b^2} \right) + g_{44}z \left( \frac{d'}{d} + t \frac{d''}{d} - t \frac{d'^2}{d^2} \right) &= 0, \\ g_{22} \left( tp' + 2p - 2pt \frac{a'}{a} \right) + tpg'_{22} &= 0, \\ g_{33} \left( tp' + 2p - 2pt \frac{b'}{b} \right) + tpg'_{33} + px \partial_x g_{33} \left( 1 - t \frac{a'}{a} \right) &= 0, \\ g_{34} \left( tp' + 2p - 2pt \left( \frac{b'}{b} + \frac{d'}{d} \right) \right) + tpg'_{34} + px \partial_x g_{34} & \\ \times \left( 1 - t \frac{a'}{a} \right) &= 0, \\ g_{44} \left( tp' + 2p - 2pt \frac{d'}{d} \right) + tpg'_{44} + px \partial_x g_{44} \left( 1 - t \frac{a'}{a} \right) &= 0. \end{aligned} \quad (\text{A.2})$$

As we can see, actually, the only ODEs that must by satisfied are:

$$\rho' t + 2\rho = 0, \quad (p't + 2p) = 0, \quad (\text{A.3})$$

which are equivalent. Hence

$$\rho' t = -2\rho, \quad L_H \rho = \rho' t = -2\rho, \quad (\text{A.4})$$

i.e.

$$\left( \phi'' \phi' + \frac{dV(\phi)}{d\phi} \phi' \right) t + 2 \left( \frac{1}{2} \phi'^2 + V(\phi) \right) = 0, \quad (\text{A.5})$$

that we may split into

$$t(\phi'' \phi') + \phi'^2 = 0, \quad t \frac{dV}{d\phi} \phi' + 2V = 0, \quad (\text{A.6})$$

where

$$t\phi'' + \phi' = 0 \Rightarrow \phi = \kappa \ln t, \quad (\text{A.7})$$

i.e.  $\mathcal{L}_H \phi' = 0$ . With regard to the second equation

$$\frac{dV}{d\phi} \kappa + 2V = 0 \Rightarrow V = K e^{-\frac{2}{\kappa} \phi}, \quad (\text{A.8})$$

i.e.  $\mathcal{L}_H V = -2V$ . Note that from Eq. (A.7),  $\phi' t = \kappa$ . We also may study the complete equation (A.5) i.e.

$$\phi'' + \frac{dV}{d\phi} + \phi' t^{-1} + 2V (t\phi')^{-1} = 0, \quad (\text{A.9})$$

under the LG method. The standard procedure brings us to get the next system of PDE:

$$t^2 \xi_{\phi\phi} = 0, \quad (\text{A.10})$$

$$t^2 \eta_{\phi\phi} - 2t^2 \xi_{t\phi} + 2t \xi_{\phi} = 0, \quad (\text{A.11})$$

$$2t^2 \eta_{t\phi} - t^2 \xi_{tt} + t \xi_t + 3t^2 \xi_{\phi} \frac{dV}{d\phi} - \xi = 0, \quad (\text{A.12})$$

$$\begin{aligned} t^2 \eta_{tt} + 8t \xi_{\phi} V + 2t^2 \xi_t \frac{dV}{d\phi} - 2t^2 \eta_{\phi} \frac{dV}{d\phi} + t \eta_t \\ + \eta t^2 \frac{d^2 V}{d\phi^2} = 0, \end{aligned} \quad (\text{A.13})$$

$$-2\xi V + 6t \xi_t V + 2t \eta \frac{dV}{d\phi} - 4t \eta_{\phi} V = 0, \quad (\text{A.14})$$

$$t \eta_t V = 0. \quad (\text{A.15})$$

The symmetry,  $\xi = \alpha t, \eta = \delta$ , brings us to obtain the following restriction on the potential (from Eq. (A.13) and (A.14))

$$2 \frac{dV}{d\phi} + \frac{d^2 V}{d\phi^2} = 0, \quad 2V + \frac{dV}{d\phi} = 0, \quad (\text{A.16})$$

and therefore we obtain as solution

$$V = \exp(-2\phi) \quad \phi = \ln t. \quad (\text{A.17})$$

Therefore we may state the following theorem.

**Theorem 4** *The only possible form for the potential  $V(\phi)$  for a spacetime admitting a HFV, HO is  $V(\phi) = V_0 \exp(\kappa\phi)$  and therefore  $\phi = \ln t$ .*

Sometimes it is interesting to study the symmetries of the tensor  $T_i^j \in \mathcal{T}_1^1(M)$ . In this case the matter collineation equations read  $\mathcal{L}_{HO}T_i^j = 0$ , iff  $\rho' = 0$ , and  $p' = 0$ , which is equivalent to

$$\phi'' = \pm \frac{dV(\phi)}{d\phi}, \quad (\text{A.18})$$

where as we can see this approach is related with the variational symmetries.

In this case the solution of Eq. (A.18) is the following one

$$t = \int^\phi \pm \frac{da}{\sqrt{-2V(a) + C_1}} + C_2. \quad (\text{A.19})$$

The Lie group methods applied to Eq. (A.18) gives

$$\xi_{\phi\phi} = 0, \quad (\text{A.20})$$

$$\eta_{\phi\phi} - 2\xi_{t\phi} = 0, \quad (\text{A.21})$$

$$2\eta_{t\phi} - \xi_{tt} + 3\xi_\phi \frac{dV}{d\phi} = 0, \quad (\text{A.22})$$

$$\eta_{tt} + 2\xi_t \frac{dV}{d\phi} - \eta_\phi \frac{dV}{d\phi} + \eta \frac{d^2V}{d\phi^2} = 0, \quad (\text{A.23})$$

where, for example, the symmetry  $\xi = t, \eta = 1$  brings us to obtain, from Eq. (A.23), the following restriction on the potential  $V$

$$2 \frac{dV}{d\phi} + \frac{d^2V}{d\phi^2} = 0, \quad (\text{A.24})$$

i.e. a solution like this:  $V = \exp(-2\phi)$ , and therefore,  $\phi = \ln t$ .

## B Matter collineation for the scalar model with $G(t)$

If the stress-energy tensor stand for a scalar model then it takes the following form:

$$T_{ij} = (\rho + p) u_i u_j + p g_{ij}, \quad (\text{B.1})$$

where

$$\rho = \frac{1}{2} \phi'^2 + V(\phi), \quad p = \frac{1}{2} \phi'^2 - V(\phi).$$

Then Eq.

$$\mathcal{L}_{HO}(G(t)T_{ij}) = 0,$$

reads

$$\frac{G'}{G} + \frac{\rho'}{\rho} = -\frac{2}{t} \Leftrightarrow G\rho \approx t^{-2},$$

now reads

$$\frac{\rho'}{\rho} = -\left(\frac{2}{t} + \frac{G'}{G}\right),$$

i.e.

$$\phi'' = -\frac{dV(\phi)}{d\phi} - \left(\frac{2}{t} + \frac{G'}{G}\right) \left(\frac{1}{2}\phi' + \frac{V(\phi)}{\phi'}\right).$$

We may follow different tactics. The first one consists in studying the whole equation

$$\phi'' = -\frac{dV(\phi)}{d\phi} - \left(\frac{2}{t} + \frac{G'}{G}\right) \left(\frac{1}{2}\phi' + \frac{V(\phi)}{\phi'}\right). \quad (\text{B.2})$$

The second one will consist in splitting the ODE in the following form (as in the standard case)

$$\phi'' = -\left(\frac{1}{t} + \frac{G'}{2G}\right) \phi', \quad (\text{B.3})$$

$$\frac{dV(\phi)}{d\phi} = -\left(\frac{2}{t} + \frac{G'}{G}\right) \frac{V(\phi)}{\phi'}, \quad (\text{B.4})$$

in such a way that solving (B.3) then we will be able to integrate (B.4).

Eq. (B.3) has the following solution

$$\phi = C_1 \int \frac{dt}{t\sqrt{G(t)}} + C_2. \quad (\text{B.5})$$

In the same way Eq. (B.3) admits the following symmetries:

$$\begin{aligned} \xi_{\phi\phi} &= 0, \\ 2\left(\frac{1}{t} + \frac{G'}{2G}\right) \xi_\phi + \eta_{\phi\phi} - 2\xi_{t\phi} &= 0, \\ \left(\frac{1}{t} + \frac{G'}{2G}\right) \xi_t + \left(-\frac{1}{t^2} + \frac{G''}{2G} - \frac{G'^2}{2G^2}\right) \xi + \eta_{t\phi} - 2\xi_{tt} &= 0, \\ \left(\frac{1}{t} + \frac{G'}{2G}\right) \eta_t + \eta_{tt} &= 0, \end{aligned}$$

where the symmetry

$$\xi = t, \quad \eta = -b\phi \Rightarrow \phi = \phi_0 t^{-b}, \quad (\text{B.6})$$

brings us to obtain the following constrain on function  $G(t)$  :

$$G'' = \frac{G'^2}{G} - \frac{G'}{t}, \quad (\text{B.7})$$

whose solution is

$$G = G_0 t^k, \quad k \in \mathbb{R}. \quad (\text{B.8})$$

The main quantities behave as follows

From

$$G\rho \approx t^{-2} \Rightarrow G\phi'^2 \approx t^{-2} \Leftrightarrow k = 2b. \quad (\text{B.9})$$

$$\begin{aligned} \phi &= \phi_0 t^{-b}, \quad G = G_0 t^{2b}, \quad V = V_0 t^{-2(b+1)}, \\ H &= h t^{-1}, \quad h \in \mathbb{R}. \end{aligned} \quad (\text{B.10})$$

Now, Eq. (B.4) yields

$$\frac{dV(\phi)}{d\phi} \frac{\phi'}{V(\phi)} = -2(1+b)t^{-1},$$

whose integration gives

$$\ln V = -2(1+b) \ln t \Leftrightarrow V = t^{-2b-2},$$

such that Eq. (B.2) is verified and therefore

$$V = \phi^\alpha = (t^{-\alpha b}) = t^{-2b-2} \Leftrightarrow \alpha = \frac{2}{b}(b+1).$$

In the same way we also may study the following equation

$$\phi'' = -\frac{dV(\phi)}{d\phi} - \left(\frac{2}{t} + \frac{G'}{G}\right) \left(\frac{1}{2}\phi' + \frac{V(\phi)}{\phi'}\right), \quad (\text{B.11})$$

through the Lie group method. Eq. (B.11) admits the following symmetries:

$$\begin{aligned} \xi_{\phi\phi} &= 0, \\ 2\left(\frac{2}{t} + \frac{G'}{G}\right) \xi_\phi + 2\eta_{\phi\phi} - 4\xi_{t\phi} &= 0, \\ 6V_\phi \xi_\phi + \left(\frac{2}{t} + \frac{G'}{G}\right) \xi_t + \left(-\frac{2}{t^2} + \frac{G''}{G} - \frac{G'^2}{G^2}\right) \xi + 4\eta_{t\phi} - 2\xi_{tt} &= 0, \\ 8\left(\frac{2}{t} + \frac{G'}{G}\right) V \xi_\phi + 4V_\phi \xi_t - 2V_\phi \eta_\phi + \left(\frac{2}{t} + \frac{G'}{G}\right) \eta_t + 2V_{\phi\phi} \eta + 2\eta_{tt} &= 0, \\ \left(\frac{2}{t} + \frac{G'}{G}\right) V_\phi \eta + 3\left(\frac{2}{t} + \frac{G'}{G}\right) V \xi_t - 2\left(\frac{2}{t} + \frac{G'}{G}\right) V \eta_\phi - \left(-\frac{2}{t^2} + \frac{G''}{G} - \frac{G'^2}{G^2}\right) V \xi &= 0, \\ \left(\frac{1}{t} + \frac{G'}{2G}\right) V \eta_t &= 0. \end{aligned}$$

As above, the symmetry

$$\xi = t, \quad \eta = -b\phi \quad \implies \quad \phi = \phi_0 t^{-b}, \quad (\text{B.12})$$

brings us to obtain the following constrain on function  $G(t)$  :

$$\left(\frac{2}{t} + \frac{G'}{G}\right) + \left(-\frac{2}{t^2} + \frac{G''}{G} - \frac{G'^2}{G^2}\right) t = 0, \quad (\text{B.13})$$

$$4V_\phi + 2bV_\phi - 2bV_{\phi\phi} = 0, \quad (\text{B.14})$$

$$-b\left(\frac{2}{t} + \frac{G'}{G}\right) V_\phi \phi + 3\left(\frac{2}{t} + \frac{G'}{G}\right) V + 2b\left(\frac{2}{t} + \frac{G'}{G}\right) V - t\left(-\frac{2}{t^2} + \frac{G''}{G} - \frac{G'^2}{G^2}\right) V = 0. \quad (\text{B.15})$$

Eq. (B.15) may be rewritten as

$$-b\left(\frac{2}{t} + \frac{G'}{G}\right) V_\phi \phi + \left((2b+3)\left(\frac{2}{t} + \frac{G'}{G}\right) - t\left(-\frac{2}{t^2} + \frac{G''}{G} - \frac{G'^2}{G^2}\right)\right) V = 0,$$

while from Eq. (B.13) we get

$$G'' = \frac{G'^2}{G} - \frac{G'}{t}, \Rightarrow G = G_0 t^k, k \in \mathbb{R}. \quad (\text{B.16})$$

From Eq. (B.14) we get

$$V_{\phi\phi} = \frac{2+b}{b} \frac{V_\phi}{\phi} \Rightarrow V = V_0 \phi^{\frac{2}{b}(b+1)} = V_0 t^{-2(b+1)}. \quad (\text{B.17})$$

Notice that we have obtained the same results as in the splitting case.

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The main quantities behave as follows

$$\begin{aligned} \phi &= \phi_0 t^{-b}, G = G_0 t^{2b}, \\ V &= V_0 t^{-2(b+1)}, H = h t^{-1}, h \in \mathbb{R}. \end{aligned} \quad (\text{B.18})$$


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